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On M-functions and operator theory for non-self-adjoint discrete Hamiltonian systems

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Abstract

We study discrete, generally non-self-adjoint Hamiltonian systems, defining Weyl–Sims sets, which replace the classical Weyl circles, and a matrix-valued M-function on suitable cone-shaped domains in the complex plane. Furthermore, we characterise realisations of the corresponding differential operator and its adjoint, and construct their resolvents.

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1. Introduction

Weyl's celebrated 1910 paper [9] initiated what is today known as the Weyl–Titchmarsh theory of the Sturm–Liouville differential equation

$$-(py')' + qy = \lambda wy \tag{1.1}$$

with real coefficients on intervals with singular end-points. One of Weyl's results is that, for each $\lambda \in \mathbb{C}\setminus\mathbb{R}$, Eq. (1.1) has a non-trivial solution which is w-square integrable near a singular end-point. The question of how many such (linearly independent) solutions exist is connected to the number of self-adjoint realisations of the corresponding Sturm–Liouville differential operator, and is answered in Weyl's alternative, leading to a general classification of singular Sturm–Liouville problems. The Titchmarsh–Weyl m-function, an analytic function in the complex upper half-plane used to characterise the distinguished solutions, plays an important role in the spectral analysis of Sturm–Liouville operators.

Sims [7] studied Eq. (1.1) with a complex-valued function q, with the aim of establishing an analogue of the Weyl–Titchmarsh theory for non-self-adjoint equations. Here the m-function is defined on a collection of rotated

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half-planes which do not intersect the numerical range of the operator. This approach was taken up in [3,4], extended to Hamiltonian systems of the form

$$Jy' = (\lambda A + B)y; \tag{1.2}$$

here A and B are complex $2n \times 2n$ matrix-valued functions with $A \geqslant 0$, and $J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$, where I_n and 0_n denote the $n \times n$ identity and null matrices, respectively. They give a classification of cases depending on the number of linearly independent A-square integrable solutions, which roughly corresponds to Weyl's alternative. They also establish the analytical properties of the matrix-valued Titchmarsh–Weyl function $M(\lambda)$, which is now defined on a collection of suitable cone-shaped regions in the complex plane, and discuss the associated differential operator and its adjoint.

Difference equations arise naturally as discretised analogues of differential equations, and appear in their own right e.g., in the recurrence formulae for special functions and orthogonal polynomials. In spectral theory, difference operators are studied as models which avoid the inherent unboundedness of differential operators, the most prominent example being Jacobi matrices, with three-term recurrence formulae as eigenvalue equations, for which a Weyl-type theory was developed by Nevanlinna and Hellinger (cf. the detailed account in [1,2]).

Clark and Gesztesy [5], motivated by a remark by Krall that, in spite of the vast literature on the subject, a Weyl–Titchmarsh theory for general discrete Hamiltonian systems was still missing, investigated the solutions, Green's functions and matrix-valued M-functions for self-adjoint boundary value problems for such systems with one or two singular end-points.

In the present paper, we follow the example of [4] to study the discrete non-self-adjoint Hamiltonian system

$$J\Delta y_k = (\lambda A_k + B_k)y_k \quad (k \in \mathbb{N}_0), \tag{1.3}$$

where $\varDelta$ is a mixed right-/left-difference operator and $(A_k)_{k\in\mathbb{N}_0}$, $(B_k)_{k\in\mathbb{N}_0}$ are $2n \times 2n$ matrix-valued sequences. Although large parts of the reasoning in [4] can be transferred to this situation, it turns out that the discrete problem presents a number of specific difficulties; thus, for example, the distinction between left and right differences has no analogue in the differential equation case, and the product rule for differences is not nearly as convenient as the corresponding rule for derivatives. We therefore put the emphasis on questions which require a different treatment than in the case of (1.2), referring to [4] for those points which carry over in a more straightforward way.

The paper is structured as follows. In Section 2, we study the properties of (1.3) as a difference equation, singling out a fundamental system of solutions on which subsequent constructions are based, and providing an analogue for the important tool of integration by parts. Section 3 introduces the Weyl–Sims sets $D_k(\lambda)$, nested sets of matrices which replace the usual Weyl circles. The elements of their limit set $D_\infty(\lambda)$ parametrise the Weyl solutions. In Section 4, we prove the existence of Titchmarsh–Weyl M-functions for our system on suitable cone-shaped regions in the complex plane. The Weyl solutions are shown to satisfy a limiting condition at infinity, which can be used to define operator domains for the difference operator L_μ associated with (1.3) and its formal adjoint $\widetilde{L}_\mu$. If $D_\infty(\mu)$ has only one element for some μ, then the function M and the operators L_μ, $\widetilde{L}_\mu$ are uniquely determined. In Section 5, we solve the inhomogeneous equation to construct the resolvent operator, and show that $\widetilde{L}_\mu$ is indeed the adjoint of L_μ.

Throughout the paper, we use the following notation. The basic difference operator in (1.3) is

$$\varDelta = \begin{pmatrix} \varDelta_+ & 0_n \\ 0_n & \varDelta_- \end{pmatrix},$$

where $\varDelta_+(\varDelta_-)$ denotes the right- (left-) difference operator for $\mathbb{C}^n$-valued sequences, $(\varDelta_+ u)_k = u_{k+1} - u_k$, $(\varDelta_- u)_k = u_k - u_{k-1}$. For each $k \in \mathbb{N}_0$, A_k and B_k are complex $2n \times 2n$ matrices with block structure

$$A_k = \begin{pmatrix} A_k^{(1)} & 0_n \\ 0_n & A_k^{(2)} \end{pmatrix}, \quad B_k = \begin{pmatrix} B_k^{(1)} & B_k^{(2)} \\ B_k^{(3)} & B_k^{(4)} \end{pmatrix}.$$

We assume that $A_k > 0$ for all $k \in \mathbb{N}_0$, but B_k is a general (not necessarily Hermitian) matrix.

A sequence

$$y = \left(\begin{pmatrix} \\ v_{-1} \end{pmatrix}, \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \dots \right)$$

is called *solution* of the difference equation (1.3) if (1.3) holds for all $k \in \mathbb{N}_0$.

We call a sequence $y=\{y_k\}_{k=k_0}^{\infty}$, $y_k\in\mathbb{C}^{2n}$, *A-square summable* if and only if $\sum_{k=0}^{\infty} y_k^* A_k y_k<\infty$. The complex Hilbert space of all A-square summable sequences is denoted by $\mathscr{H}_A$, with scalar product

$$(f,g)_A=\sum_{k=0}^{\infty} g_k^* A_k f_k$$

and norm $\|f\|_A=(f,f)_A^{1/2}(f,g\in\mathscr{H}_A)$. Moreover, a $2n\times n$ matrix is called *A-square summable* if and only if each of its columns is A-square summable. We use a corresponding terminology for other $2n\times 2n$ weight matrices.

2. The homogeneous difference equation

In this section we study the basic properties of the discrete Hamiltonian system (1.3) as a difference equation, noting the existence and uniqueness of solutions of initial-value problems and defining a canonical fundamental system which will serve as a basis for the subsequent constructions. A certain complication arises from the distinction between left and right differences in (1.3); this becomes most prominent in the analogue of integration by parts (Theorem 2.3), where the sequences appear shifted in the lower half on the right-hand side. For solutions of the Hamiltonian system and its adjoint, this partial shift can be conveniently expressed as multiplication with a suitable matrix function (see (2.1), (2.5)).

Lemma 2.1 (*Existence and Uniqueness Theorem for (1.3)*). *Assume that $(I_n+B_k^{(2)})^{-1}$ and $(I_n+B_k^{(3)})^{-1}$ exist for all $k\in\mathbb{N}_0$, and suppose that u_K and v_K are given for some $K\in\mathbb{N}_0$. Then there exists a unique solution y of Eq.* (1.3) *such that $y_k=\binom{u_k}{v_k}$.*

This can be easily seen by writing (1.3) in the form

$$u_{k+1}=(I_n+B_k^{(3)})u_k+(\lambda A_k^{(2)}+B_k^{(4)})v_k,$$
$$v_{k+1}=(I_n+B_{k+1}^{(2)})^{-1}(-(\lambda A_{k+1}^{(1)}+B_{k+1}^{(1)})u_{k+1}+v_k).$$

Lemma 2.2. *Suppose that $(I_n+B_k^{(2)})^{-1}$ exists for all $k\in\mathbb{N}_0$ and that u_0 is given. Then there exists a unique solution y of Eq.* (1.3) *such that $v_{-1}=0$ and u_0 is the given value. This solution can be interpreted as satisfying the boundary condition at* 0,

$$(I_n+B_0^{(2)})v_0=-(\lambda A_0^{(1)}+B_0^{(1)})u_0.$$

Theorem 2.3 (*Summation by parts*). *Let*

$$y=\left(\begin{pmatrix} \\ v_{-1}\end{pmatrix},\begin{pmatrix}u_0\\ v_0\end{pmatrix},\begin{pmatrix}u_1\\ v_1\end{pmatrix},\ldots\right)$$

and

$$z=\left(\begin{pmatrix} \\ \widetilde{v}_{-1}\end{pmatrix},\begin{pmatrix}\widetilde{u}_0\\ \widetilde{v}_0\end{pmatrix},\begin{pmatrix}\widetilde{u}_1\\ \widetilde{v}_1\end{pmatrix},\ldots\right).$$

Then for $m\in\mathbb{N}_0$, $k>m$ we have

$$\sum_{i=m}^{k-1}[z_k^*(J\Delta y_k)-(J\Delta z_k)^* y_k]=\widehat{z}_k^* J\widehat{y}_k-\widehat{z}_m^* J\widehat{y}_m,$$

where

$$\widehat{y}_k=\begin{pmatrix}u_k\\ v_{k-1}\end{pmatrix}\quad \text{and}\quad \widehat{z}_k=\begin{pmatrix}\widetilde{u}_k\\ \widetilde{v}_{k-1}\end{pmatrix}\quad (k\in\mathbb{N}_0).$$

Proof.

$$\sum_{i=m}^{k-1}[z_i^*(J\Delta y_i)-(J\Delta z_i)^*y_i]=\sum_{i=m}^{k-1}(\widetilde{u}_i^*v_{i-1}-\widetilde{u}_{i+1}^*v_i+\widetilde{v}_i^*u_{i+1}-\widetilde{v}_{i-1}^*u_i)$$
$$=\sum_{i=m}^{k-1}[\Delta_+(\widetilde{v}_{i-1}^*u_i)-\Delta_+(\widetilde{u}_i^*v_{i-1})]$$
$$=[\widetilde{v}_{i-1}^*u_i-\widetilde{u}_i^*v_{i-1}]_{i=m}^k=\widehat{z}_i^*J\widehat{y}_i|_{i=m}^k. \qquad \square$$

If y is a solution of Eq. (1.3), it is related to $\widehat{y}$ via

$$y_k=H_k\widehat{y}_k \quad (k\in\mathbb{N}_0), \tag{2.1}$$

where

$$H_k=\begin{pmatrix} I_n & 0_n \\ -(I_n+B_k^{(2)})^{-1}(\lambda A_k^{(1)}+B_k^{(1)}) & (I_n+B_k^{(2)})^{-1}\end{pmatrix} \tag{2.2}$$

($k\in\mathbb{N}_0$). Note that H_k is regular with

$$H_k^{-1}=\begin{pmatrix} I_n & 0_n \\ (\lambda A_k^{(1)}+B_k^{(1)}) & (I_n+B_k^{(2)})\end{pmatrix}, \quad k\in\mathbb{N}_0.$$

Hence, Eq. (1.3) is equivalent to the right-difference equation

$$J\Delta_+\widehat{y}_k=(\lambda A_k+B_k)H_k\widehat{y}_k \quad (\lambda\in\mathbb{C}, k\in\mathbb{N}_0), \tag{2.3}$$

since $J\Delta y_k=J\Delta_+\widehat{y}_k$. Clearly an Existence and Uniqueness Theorem analogous to Lemma 2.1 holds for Eq. (2.3).

The formal adjoint of Eq. (1.3) takes the form

$$J\Delta z_k=(\bar{\lambda}A_k+B_k^*)z_k \quad (\lambda\in\mathbb{C}, k\in\mathbb{N}_0); \tag{2.4}$$

an Existence and Uniqueness Theorem for Eq. (2.4) holds under the hypotheses of Lemma 2.1.

A solution z of Eq. (2.4) is related to $\widehat{z}$ via

$$z_k=\widetilde{H}_k\widehat{z}_{k+1} \quad (k\in\mathbb{N}_0), \tag{2.5}$$

where

$$\widetilde{H}_k=\begin{pmatrix} (I_n+B_k^{(2)*})^{-1} & -(I_n+B_k^{(2)*})^{-1}(\bar{\lambda}A_k^{(2)}+B_k^{(4)*}) \\ 0_n & I_n\end{pmatrix} \quad (k\in\mathbb{N}_0).$$

Note that

$$\widetilde{H}_k^{-1}=\begin{pmatrix} (I_n+B_k^{(2)*}) & (\bar{\lambda}A_k^{(2)}+B_k^{(4)*}) \\ 0_n & I_n\end{pmatrix} \quad (k\in\mathbb{N}_0).$$

Hence, Eq. (2.4) is equivalent to the equation

$$J\Delta_-\widehat{z}_{k+1}=J\Delta_+\widehat{z}_k=(\bar{\lambda}A_k+B_k^*)\widetilde{H}_k\widehat{z}_{k+1}=H_k^*(\bar{\lambda}A_k+B_k^*)\widehat{z}_{k+1} \tag{2.6}$$

($\lambda\in\mathbb{C},\ k\in\mathbb{N}_0$). A sequence $(\widehat{Y}_k)_{k\in\mathbb{N}_0}$ of $2n\times 2n$ matrices is called a *fundamental system* of (2.3) if its columns are linearly independent solutions of Eq. (2.3); clearly this is the case if and only if $\widehat{Y}_k$ has rank $2n$ for some $k\in\mathbb{N}_0$. In this case the columns of $Y_k=H_k\widehat{Y}_k$ form a fundamental system (system of linearly independent solutions) of (1.3). In the following, let the $\mathbb{C}^{2n,2n}$ matrix sequence $\widehat{Y}=((\widehat{\theta}_k|\widehat{\phi}_k))_{k\in\mathbb{N}_0}$ be the fundamental system of (2.3) satisfying the initial condition

$$\widehat{Y}_0=J. \tag{2.7}$$

Then $Y = ((\theta_k|\phi_k))_{k\in\mathbb{N}_0}$ is the fundamental system of (1.3) satisfying

$$Y_0 = \begin{pmatrix} 0_n & -I_n \\ (I_n + B_0^{(2)})^{-1} & (I_n + B_0^{(2)})^{-1}(\lambda A_0^{(1)} + B_0^{(1)}) \end{pmatrix}.$$

Similarly, let $\widehat{Z} = ((\widehat{\eta}_k|\widehat{\chi}_k))_{k\in\mathbb{N}_0}$ be the fundamental system of (2.6) satisfying $\widehat{Z}_0 = J$, then

$$Z_0 = \begin{pmatrix} 0_n & -I_n \\ (I_n + B_0^{(3)*})^{-1} & (I_n + B_0^{(3)*})^{-1}(\overline{\lambda} A_0^{(1)} + B_0^{(1)*}) \end{pmatrix}.$$

Lemma 2.4. *For the above solutions $\widehat{Y}$, $\widehat{Z}$,*

$$\widehat{Z}_k = -J(\widehat{Y}_k^{-1})^* J \quad (k \in \mathbb{N}_0)$$

holds.

Proof. Let $\widehat{U}_k := -J(\widehat{Y}_k^{-1})^* J$. Since $\widehat{Y}_0 = J$, it follows that

$$\widehat{U}_0 = -J(\widehat{Y}_0^{-1})^* J = J = \widehat{Z}_0.$$

We now show that $\widehat{U}$ is a solution of (2.6), so $\widehat{U} = \widehat{Z}$ by uniqueness. As

$$0 = \Delta_+(\widehat{Y}_k^{-1}\widehat{Y}_k) = \widehat{Y}_{k+1}^{-1}(\Delta_+\widehat{Y}_k) + (\Delta_+\widehat{Y}_k^{-1})\widehat{Y}_k,$$

Eq. (2.3) implies

$$\Delta_+(\widehat{Y}_k^{-1}) = -\widehat{Y}_{k+1}^{-1}(\Delta_+\widehat{Y}_k)\widehat{Y}_k^{-1} = \widehat{Y}_{k+1}^{-1} J(\lambda A_k + B_k)H_k.$$

Hence,

$$\begin{aligned} J\Delta_+\widehat{U}_k &= J\Delta_+[-J(\widehat{Y}_k^{-1})^* J] = \Delta_+(\widehat{Y}_k^{-1})^* J \\ &= [\widehat{Y}_{k+1}^{-1} J(\lambda A_k + B_k)H_k]^* J = H_k^*(\overline{\lambda} A_k + B_k^*)\widehat{U}_{k+1}. \qquad \Box \end{aligned}$$

3. Weyl–Sims nesting sets

In this section we introduce the Weyl–Sims sets $D_k(\lambda)$, $k \in \mathbb{N}_0$, the analogue of the classical Weyl circles. The spectral parameter λ varies in a set $\Lambda(\lambda_0, \mathcal{U}_{2n}) \subset \mathbb{C}$, a cone-shaped set defined in analogy to the construction of [4], which takes the role of Sims' rotated half-planes. Here, $\mathcal{U}_{2n}$ is any one of a large class of matrices describing the rotation. The central observation is the nesting property of the Weyl–Sims sets (Theorem 3.2). As a consequence, there is a limit set $D_\infty(\lambda)$ with the property that for any $l \in D_\infty(\lambda)$, the Weyl solution $\psi = \theta + \phi l$ is square summable with respect to a certain weight function W (an analogous statement holds for the adjoint equation). We conclude this section by noting conditions which imply A-square summability of the Weyl solution.

We choose $\mathcal{U} \in \mathbb{C}^{n,n}$ regular and define

$$\mathcal{U}_{2n} = \begin{pmatrix} \mathcal{U} & 0_n \\ 0_n & -\mathcal{U}^* \end{pmatrix}.$$

Then $\mathcal{U}_{2n} J$ is Hermitian and has exactly n positive and exactly n negative eigenvalues. Indeed, if υ is an eigenvector of $\mathcal{U}_{2n} J$ with eigenvalue λ, then $w = \begin{pmatrix} 0_n & -\mathcal{U} \\ \mathcal{U}^* & 0_n \end{pmatrix} \upsilon$ is an eigenvector of $\mathcal{U}_{2n} J$ with eigenvalue $-\lambda$.

As $J\mathscr{U}_{2n}^* = -\mathscr{U}_{2n}J$, the fundamental system $\widehat{Y} = (\widehat{\theta}, \widehat{\phi})$ satisfies

$$\begin{aligned}\widehat{Y}_k^*\mathscr{U}_{2n}J\widehat{Y}_k - \widehat{Y}_0^*\mathscr{U}_{2n}J\widehat{Y}_0 &= \sum_{i=0}^{k-1}\{Y_i^*(\mathscr{U}_{2n}J\Delta Y_i) + (\mathscr{U}_{2n}J\Delta Y_i)^*Y_i\}\\ &= \sum_{i=0}^{k-1} Y_i^*\{\mathscr{U}_{2n}(\lambda A_i + B_i) + [\mathscr{U}_{2n}(\lambda A_i + B_i)]^*\}Y_i\\ &= 2\sum_{i=0}^{k-1} Y_i^* W_i(\lambda)Y_i, \qquad (3.1)\end{aligned}$$

where $W_k(\lambda) := \mathrm{Re}[\mathscr{U}_{2n}(\lambda A_k + B_k)]$.

Definition. Let $\lambda_0 \in \mathbb{C}$ and $\mathscr{U}_{2n}$ be as above. Then $(\lambda_0, \mathscr{U}_{2n})$ is called an *admissible pair for* Eq. (1.3) if

$$W_k(\lambda_0) = \mathrm{Re}[\mathscr{U}_{2n}(\lambda_0 A_k + B_k)] \geqslant 0 \quad (k \in \mathbb{N}_0). \qquad (3.2)$$

In this case, we define the set

$$\Lambda(\lambda_0, \mathscr{U}_{2n}) := \{\lambda \in \mathbb{C} : \text{ there is } \delta = \delta(\lambda) > 0 \text{ such that } \mathrm{Re}[(\lambda - \lambda_0)\mathscr{U}_{2n}A_k] \geqslant \delta\mathscr{U}_{2n}A_k\mathscr{U}_{2n}^*,\ (k \in \mathbb{N}_0)\}. \quad (3.3)$$

The Weyl–Sims sets for (1.3) are defined for $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ as

$$D_k(\lambda) := \left\{ l \in \mathbb{C}^{n,n} : \left(\widehat{\theta}_k + \widehat{\phi}_k l\right)^* \mathscr{U}_{2n} J \left(\widehat{\theta}_k + \widehat{\phi}_k l\right) \leqslant 0 \right\}. \qquad (3.4)$$

Before we can prove the nesting property of the Weyl–Sims sets (Theorem 3.2), we need a preparatory step. Note that

$$W_k(\lambda) \geqslant \delta\mathscr{U}_{2n}A_k\mathscr{U}_{2n}^* > 0 \quad (k \in \mathbb{N}_0,\ \lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})), \qquad (3.5)$$

so for any $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ and $\zeta \in \mathbb{C}^n$, $(W_k(\lambda)\phi_k)\zeta = 0$ $(k \in \mathbb{N}_0)$ implies that $\zeta = 0$. Setting $\widehat{\theta}_k = \begin{pmatrix}\theta_k^{(1)}\\ \theta_{k-1}^{(2)}\end{pmatrix}$ and $\widehat{\phi}_k = \begin{pmatrix}\phi_k^{(1)}\\ \phi_{k-1}^{(2)}\end{pmatrix}$ where $\theta_k^{(1)}, \theta_{k-1}^{(2)}, \phi_k^{(1)}, \phi_{k-1}^{(2)}$ are $\mathbb{C}^{n,n}$-valued matrices, we write

$$\widehat{Y}_k^*\mathscr{U}_{2n}J\widehat{Y}_k = 2\begin{pmatrix} S_k & T_k\\ T_k^* & P_k\end{pmatrix} \qquad (3.6)$$

with

$$\begin{aligned} S_k(\lambda) &= -\mathrm{Re}[\theta_k^{(1)*}\mathscr{U}\theta_{k-1}^{(2)}],\\ T_k(\lambda) &= -\tfrac{1}{2}[\theta_k^{(1)*}\mathscr{U}\phi_{k-1}^{(2)} + \theta_{k-1}^{(2)*}\mathscr{U}^*\phi_k^{(1)}],\\ P_k(\lambda) &= -\mathrm{Re}[\phi_k^{(1)*}\mathscr{U}\phi_{k-1}^{(2)}].\end{aligned}$$

The initial condition (2.7) implies that $P_0(\lambda) = 0$. The following statement can be proved as Lemma 3.5 in [4].

Lemma 3.1. *Let $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})$. Then some $k_0(\lambda) \in \mathbb{N}_0$ exists such that*

(i) *$P_k(\lambda)$ is increasing in k, $P_k(\lambda) \geqslant 0$, and, for $k \geqslant k_0(\lambda)$, $P_k(\lambda) > 0$,*
(ii) *$D_k(\lambda) \neq \emptyset$ for $k \geqslant k_0(\lambda)$.*

For $k \geqslant k_0(\lambda)$, we use the notation

$$\mathscr{C}_k(\lambda) := -(P_k^{-1}T_k^*)(\lambda), \quad \mathscr{R}_k(\lambda) := (T_k P_k^{-1} T_k^* - S_k)(\lambda).$$

Then, multiplying (3.6) by $(I_n|l^*)$ from the left and $\binom{I_n}{l}$ from the right, we find

$$(\widehat{\theta}_k + \widehat{\phi}_k l)^* \mathscr{U}_{2n} J(\widehat{\theta}_k + \widehat{\phi}_k l) = 2[l^* P_k l + T_k l + l^* T_k^* + S_k].$$

Thus,

$$D_k(\lambda) = \{l \in \mathbb{C}^{n,n} : (l - \mathscr{C}_k(\lambda))^* P_k(\lambda)(l - \mathscr{C}_k(\lambda)) \leqslant \mathscr{R}_k(\lambda)\}. \tag{3.7}$$

Following [4, Lemma 3.5(iii)], one can show that $\mathscr{R}_k(\lambda) \geqslant 0$, and $\mathscr{R}_k(\lambda) > 0$ if $k \geqslant k_0$, so

$$D_k(\lambda) = \{l \in \mathbb{C}^{n,n} : l = \mathscr{C}_k(\lambda) + P_k^{-1/2}(\lambda) V \mathscr{R}_k^{1/2}(\lambda), \quad \text{for some } V \in \mathbb{C}^{n,n}, \ V^* V \leqslant I_n\}, \tag{3.8}$$

noting that V can be obtained as $V = P_k^{1/2}(\lambda)(l - \mathscr{C}_k(\lambda))\mathscr{R}_k^{-1/2}(\lambda)$.

Theorem 3.2. *Let $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})$. Then*

(i) $D_k(\lambda) \subset D_{k-1}(\lambda)$ $(k \in \mathbb{N})$,
(ii) $D_k(\lambda)$ *is compact and convex* $(k \geqslant k_0(\lambda), \ k \in \mathbb{N})$.

Proof. On multiplying (3.1) by $(I_n|l^*)$ on the left and $\binom{I_n}{l}$ on the right, we find

$$(\widehat{\theta}_k + \widehat{\phi}_k l)^* \mathscr{U}_{2n} J(\widehat{\theta}_k + \widehat{\phi}_k l) = (\widehat{\theta}_0 + \widehat{\phi}_0 l)^* \mathscr{U}_{2n} J(\widehat{\theta}_0 + \widehat{\phi}_0 l) + 2\sum_{i=0}^{k-1} (\theta_i + \phi_i l)^* W_i(\lambda)(\theta_i + \phi_i l).$$

Let $l \in D_k(\lambda)$. Then

$$(\widehat{\theta}_0 + \widehat{\phi}_0 l)^* \mathscr{U}_{2n} J(\widehat{\theta}_0 + \widehat{\phi}_0 l) + 2\sum_{i=0}^{k-1} (\theta_i + \phi_i l)^* W_i(\lambda)(\theta_i + \phi_i l) \leqslant 0, \tag{3.9}$$

and since $W_{k-1}(\lambda) > 0$, this implies

$$(\widehat{\theta}_0 + \widehat{\phi}_0 l)^* \mathscr{U}_{2n} J(\widehat{\theta}_0 + \widehat{\phi}_0 l) + 2\sum_{i=0}^{k-2} (\theta_i + \phi_i l)^* W_i(\lambda)(\theta_i + \phi_i l) \leqslant 0.$$

Part (ii) is proved as in [4, Theorem 3.6(iii)] using the representation (3.8) of $D_k(\lambda)$. $\square$

Because of the nesting property (Theorem 3.2(i)), there exists a limiting set $D_\infty(\lambda)$, which may contain only one point. If $l \in D_\infty(\lambda)$, it follows from (3.9) that

$$\sum_{i=0}^{k-1} (\theta_i + \phi_i l)^* W_i(\lambda)(\theta_i + \phi_i l) \leqslant -\frac{1}{2}(\widehat{\theta}_0 + \widehat{\phi}_0 l)^* \mathscr{U}_{2n} J(\widehat{\theta}_0 + \widehat{\phi}_0 l)$$

for all $k \in \mathbb{N}_0$. Therefore,

$$\sum_{i=0}^{\infty} (\theta_i + \phi_i l)^* W_i(\lambda)(\theta_i + \phi_i l) < \infty$$

which means that the sequence $\underset{\sim}{\psi}(\lambda) := \theta(\lambda) + \phi(\lambda) l$ is $W(\lambda)$-square summable. By virtue of (3.5), it follows that $\psi(\lambda)$ is $\widetilde{A}$-square summable, where $\widetilde{A}_k := \mathscr{U}_{2n} A_k \mathscr{U}_{2n}^*$.

We remark that in representation (3.7) of the set $D_k(\lambda)$, $\mathscr{C}_k(\lambda)$ plays the role of the centre, $\mathscr{R}_k(\lambda)$—here a matrix—that of the radius of the Weyl circles. As in [4], one can prove that $\mathscr{R}_k(\lambda)$ is eventually decreasing (in the quadratic form sense) and that $\mathscr{C}_k(\lambda)$ converges to a limit $\mathscr{C}_\infty(\lambda) \in D_\infty(\lambda)$.

For the fundamental system $\widehat{Z}$ of the adjoint equation (2.4), we have in analogy to (3.1)

$$\widehat{Z}_k^* \mathscr{U}_{2n}^{-1} J \widehat{Z}_k |_{k=0}^n = 2 \sum_{k=0}^{n-1} Z_k^* \widetilde{W}_k(\lambda) Z_k, \tag{3.10}$$

where

$$\widetilde{W}_k(\lambda) := \mathrm{Re}[(\lambda A_k + B_k)\mathscr{U}_{2n}^{-1*}] = \mathrm{Re}[\mathscr{U}_{2n}^{-1}(\bar{\lambda} A_k + B_k^*)].$$

Note that $W_k(\lambda) = \mathscr{U}_{2n} \widetilde{W}_k(\lambda) \mathscr{U}_{2n}^*$.

We call $(\lambda_0, \mathscr{U}_{2n}^{-1})$ an admissible pair for the adjoint equation if

$$\widetilde{W}_k(\lambda_0) := \mathrm{Re}[(\lambda_0 A_k + B_k)\mathscr{U}_{2n}^{-1*}] \geqslant 0,$$

and define the set

$$\begin{aligned}\widetilde{\Lambda}(\lambda_0, \mathscr{U}_{2n}^{-1}) &:= \{\lambda \in \mathbb{C} : \text{there is } \delta > 0 \text{ such that } \mathrm{Re}[(\lambda - \lambda_0) A_k \mathscr{U}_{2n}^{-1*}] \geqslant \delta \mathscr{U}_{2n}^{-1} A_k \mathscr{U}_{2n}^{-1*} \ (k \in \mathbb{N}_0)\} \\ &= \{\lambda \in \mathbb{C} : \text{there is } \delta > 0 \text{ such that } \mathrm{Re}[(\lambda - \lambda_0)\mathscr{U}_{2n} A_k] \geqslant \delta A_k \ (k \in \mathbb{N}_0)\}.\end{aligned} \tag{3.11}$$

We remark that $(\lambda_0, \mathscr{U}_{2n})$ is an admissible pair for Eq. (1.3) if and only if $(\lambda_0, \mathscr{U}_{2n}^{-1})$ is an admissible pair for the adjoint equation. For $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n}) \cap \widetilde{\Lambda}(\lambda_0, \mathscr{U}_{2n}^{-1})$ and sufficiently large k, the Weyl–Sims sets for (2.4) can be shown as in [4] to be $\widetilde{D}_k(\lambda) = \{l^* : l \in D_k(\lambda)\} = D_k^*(\lambda)$.

Remarks. (1) For $\lambda \in \widetilde{\Lambda}(\lambda_0, \mathscr{U}_{2n}^{-1})$ it follows from (3.10) (analogous to the proof of Theorem 3.2(i)) that $l^* \in \widetilde{D}_k(\lambda)$ if and only if

$$\sum_{i=0}^{k-1} (\eta_i + \chi_i l^*)^* \widetilde{W}_i(\lambda)(\eta_i + \chi_i l^*) \leqslant -\frac{1}{2}(\widehat{\eta}_0 + \widehat{\chi}_0 l^*)^* \mathscr{U}_{2n}^{-1} J (\widehat{\eta}_0 + \widehat{\chi}_0 l^*).$$

If $l^* \in \widetilde{D}_\infty(\lambda)$, then $\zeta(\lambda) = \eta(\lambda) + \chi(\lambda) l^*$ satisfies $\sum_{k=0}^\infty \zeta_k^* \widetilde{W}_k(\lambda) \zeta_k < \infty$, which means that $\zeta = \{\zeta_k\}_{k=0}$ is $\widetilde{W}(\lambda)$-square summable.

(2) If

$$W_k(\lambda) \geqslant \delta A_k \quad (k \in \mathbb{N}_0) \quad \text{for some } \delta > 0 \text{ and } \lambda \in \mathbb{C},$$

then $\mathscr{H}_{W(\lambda)} \subseteq \mathscr{H}_A$. This condition holds in the following cases:

1. if $\lambda \in \widetilde{\Lambda}(\lambda_0, \mathscr{U}_{2n}^{-1})$, by using (3.2) and (3.11),
2. if

$$\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n}) \quad \text{and} \quad \widetilde{A}_k \geqslant \gamma A_k \quad \text{for some } \gamma > 0 \tag{3.12}$$

by using (3.2) and (3.3).

(3) If $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})$, then $\widetilde{W}_k(\lambda) \geqslant \delta A_k$ for some $\delta > 0$, and consequently $\mathscr{H}_{\widetilde{W}(\lambda)} \subseteq \mathscr{H}_A$. Thus, if $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n}) \cap \widetilde{\Lambda}(\lambda_0, \mathscr{U}_{2n}^{-1})$, then

$$\mathscr{H}_{W(\lambda)} \cup \mathscr{H}_{\widetilde{W}(\lambda)} \subseteq \mathscr{H}_A$$

and ψ, ζ are A-square summable.

(4) Condition (3.12) implies, by (3.3), (3.11), that

$$\Lambda(\lambda_0, \mathscr{U}_{2n}) \subseteq \widetilde{\Lambda}(\lambda_0, \mathscr{U}_{2n}^{-1}) \tag{3.13}$$

for $(\lambda_0, \mathscr{U}_{2n}) \in S$. If, in addition to (3.12), a reverse inequality $A_k \geqslant \widetilde{\gamma} \widetilde{A}_k$ holds for some $\widetilde{\gamma} > 0$, i.e., if $\widetilde{A}_k \asymp A_k$, then equality holds in (3.13).

(5) As in [4], the structure of the shifted limit set $D_\infty(\lambda)-\mathscr{C}_\infty(\lambda)$ gives information about the number of $W(\lambda)$-square summable solutions. More precisely, let $\mathscr{N}(\lambda) := \bigcup_{N\in D_\infty(\lambda)}\operatorname{range}(N-\mathscr{C}_\infty(\lambda))$ and r the dimension of the linear hull of $\mathscr{N}(\lambda)$. Then there are at least $n+r$ linearly independent $W(\lambda)$-square summable solutions of (1.3), and if $\mathscr{R}_k \nrightarrow 0$ $(k\to\infty)$, the number is exactly $n+r$.

Moreover, if $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n}) \cap \widetilde{\Lambda}(\lambda_0, \mathscr{U}_{2n}^{-1})$, then (2.4) has exactly n linearly independent $\widetilde{W}(\lambda)$-square summable solutions if $\mathscr{R}_k \to 0$ $(k\to\infty)$. If $r=0$, then also $\widetilde{r}=0$ ($\widetilde{r}$ being the corresponding number for the adjoint equation) and at least one of Eqs. (1.3), (2.4) has exactly n linearly independent solutions which are $W(\lambda)$, $\widetilde{W}(\lambda)$-square summable, respectively.

4. Definition of the operators L_μ and $\widetilde{L}_\mu$

We prove in Theorem 4.1 that, for fixed $\mu \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ and $M_0 \in D_\infty(\mu)$, there exists a matrix-valued function $M(\lambda)$ $(\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n}))$ such that $M(\mu)=M_0$. Moreover, the Weyl solutions satisfy condition (4.3) at infinity. This observation plays a significant role in the definition of the operator L_μ associated with (1.3). Similarly, we define the operator $\widetilde{L}_\mu$ corresponding to the adjoint problem.

Hereafter we shall assume that

$$\widetilde{A}_k = \mathscr{U}_{2n} A_k \mathscr{U}_{2n}^* \asymp A_k. \tag{4.1}$$

This is the case e.g., if $A_k^{(j)} = a_k^{(j)} I_n + \widetilde{A}_k^{(j)}$ with $a_k^{(j)}$ arbitrary and $1/c \leqslant \widetilde{A}_k^{(j)} \leqslant c$ for some constant $c>1$ $(k\in\mathbb{N}_0,\ j\in\{1,2\})$.

Theorem 4.1. *Let $\mu \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ and $M_0 \in D_\infty(\mu)$ be fixed. Then there exists a function $M : \Lambda(\lambda_0, \mathscr{U}_{2n}) \to \mathbb{C}^{n,n}$ such that $M(\mu)=M_0$, $M(\lambda)\in D_\infty(\lambda)$ and*

$$M(\lambda)-M_0 = (\lambda-\mu)\sum_{k=0}^{\infty} \zeta_k^*(\mu) A_k \psi_k(\lambda) = (\lambda-\mu)\sum_{k=0}^{\infty} \zeta_k^*(\lambda) A_k \psi_k(\mu) \quad (\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})). \tag{4.2}$$

Here $\psi_k(\lambda) := \theta_k(\lambda) + \phi_k(\lambda) M(\lambda)$ and $\zeta_k(\lambda) := \eta_k(\lambda) + \chi_k(\lambda) M(\lambda)^*$.

Moreover,

$$\lim_{k\to\infty} \widehat{\zeta}_k^*(\mu) J \widehat{\psi}_k(\lambda) = \lim_{k\to\infty} \widehat{\zeta}_k^*(\lambda) J \widehat{\psi}_k(\mu) = 0. \tag{4.3}$$

Remark. Clearly, $M(\lambda)$ has only one possible value if $D_\infty(\lambda)$ has only one element. Consequently, if there is at least one point $\mu \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ such that $D_\infty(\lambda)$ has only one element, the function M is uniquely determined.

Proof of Theorem 4.1. Let $\psi(\mu) := \theta(\mu) + \phi(\mu) M_0$ and $\zeta(\mu) := \eta(\mu) + \chi(\mu) M_0^*$. For all $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ and $K \geqslant k_0(\lambda)$, the matrix $\widehat{\zeta}_K^*(\mu) J \widehat{\phi}_K(\lambda)$ is invertible, as one can see following the proof of Lemma 5.3 of [4]. Consequently,

$$l_K(\lambda) := -[\widehat{\zeta}_K^*(\mu) J \widehat{\phi}_K(\lambda)]^{-1} [\widehat{\zeta}_K^*(\mu) J \widehat{\theta}_K(\lambda)] \tag{4.4}$$

is well defined for $K \geqslant k_0(\lambda)$, and

$$\psi_k(K,\lambda) := \theta_k(\lambda) + \phi_k(\lambda) l_K(\lambda) \quad (k\in\mathbb{N}_0) \tag{4.5}$$

satisfies the condition

$$\widehat{\zeta}_K^*(\mu) J \widehat{\psi}_K(K,\lambda) = 0_n. \tag{4.6}$$

Moreover,

$$\widehat{\zeta}_K^*(\mu) J \widehat{\psi}_K(\mu) = (I_n | M_0) \widehat{Z}_K^*(\mu) J \widehat{Y}_K(\mu) \begin{pmatrix} I_n \\ M_0 \end{pmatrix} = 0_n \tag{4.7}$$

because of Lemma 2.4. All three matrices $\widehat{\zeta}_K(\mu)$, $\widehat{\psi}_K(\mu)$, $\widehat{\psi}_K(K,\lambda)$ have rank n, hence (4.6) and (4.7) imply that the ranges of $\widehat{\psi}_K(\mu)$ and $\widehat{\psi}_K(K,\lambda)$ coincide, i.e., that $\widehat{\psi}_K(K,\lambda) = \widehat{\psi}_K(\mu)\Omega$ for some invertible $\Omega \in \mathbb{C}^{n,n}$. This yields

$$\widehat{\psi}_K^*(K,\lambda)\mathscr{U}_{2n}J\widehat{\psi}_K(K,\lambda) = \Omega^*[\widehat{\psi}_K^*(\mu)\mathscr{U}_{2n}J\widehat{\psi}_K(\mu)]\Omega. \tag{4.8}$$

By (3.4), the matrix in brackets on the right-hand side in (4.8) is non-positive, since $M_0 \in D_\infty(\mu) \subset D_K(\mu)$ for all $K \geqslant k_0$. Thus, the left-hand side of (4.8) is non-positive, so again by (3.4)

$$l_K(\lambda) \in D_K(\lambda). \tag{4.9}$$

Define in analogy to (4.5) for $K \geqslant k_0(\lambda)$, $\zeta_k(K,\lambda) := \eta_k(\lambda) + \chi_k(\lambda)l_K(\lambda)^*$ $(k \in \mathbb{N}_0)$. Then, again using Lemma 2.4,

$$\widehat{\zeta}_K^*(K,\lambda)J\widehat{\psi}_K(K,\lambda) = (I_n|l_K(\lambda))\widehat{Z}_K^*(\lambda)J\widehat{Y}_K(\lambda)\begin{pmatrix} I_n \\ l_K(\lambda)\end{pmatrix} = 0_n. \tag{4.10}$$

Since all three matrices $\widehat{\zeta}_K(K,\lambda)$, $\widehat{\psi}_K(K,\lambda)$, $\widehat{\zeta}_K(\mu)$ have rank n, (4.6) and (4.10) show that the ranges of $\widehat{\zeta}_K(K,\lambda)$ and $\widehat{\zeta}_K(\mu)$ coincide. In particular, the (n-dimensional) range of $\widehat{\zeta}_K(K,\lambda)$ is independent of λ (as long as $k_0(\lambda) \leqslant K$), whence (4.10) implies $\widehat{\zeta}_K^*(K,\widetilde{\lambda})J\widehat{\psi}_K(K,\lambda) = 0_n$ for all $\lambda, \widetilde{\lambda} \in \Lambda(\lambda_0, \mathscr{U}_{2n})$, $K \geqslant \max\{k_0(\lambda), k_0(\widetilde{\lambda})\}$. Applying Theorem 2.3 to $\zeta_k(K,\widetilde{\lambda})$ and $\psi_k(K,\lambda)$, we find

$$\begin{aligned} l_K(\lambda) - l_K(\widetilde{\lambda}) &= \widehat{\zeta}_0^*(K,\widetilde{\lambda})J\widehat{\psi}_0(K,\lambda) - \widehat{\zeta}_K^*(K,\widetilde{\lambda})J\widehat{\psi}_K(K,\lambda) \\ &= (\lambda - \widetilde{\lambda})\sum_{k=0}^{K-1} \zeta_k^*(K,\widetilde{\lambda})A_k\psi_k(K,\lambda). \end{aligned} \tag{4.11}$$

Taking $\widetilde{\lambda} := \mu$ and noting that $l_K(\mu) = M_0$ by Lemma 2.4 and (4.4), so that $\psi_k(K,\mu) = \psi_k(\mu)$ and $\zeta_k(K,\mu) = \zeta_k(\mu)$, we obtain

$$l_K(\lambda) - M_0 = (\lambda - \mu)\sum_{k=0}^{K-1} \zeta_k^*(\mu)A_k\psi_k(K,\lambda) = (\lambda - \mu)\sum_{k=0}^{K-1} \zeta_k^*(K,\lambda)A_k\psi_k(\mu) \tag{4.12}$$

for each $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ and all $K \geqslant \max\{k_0(\lambda), k_0(\mu)\}$. (For the second identity we have interchanged the roles of λ and $\widetilde{\lambda}$ in (4.11).)

Now keeping λ fixed in the following, we try to achieve convergence in (4.12) as $K \to \infty$. Define

$$\widetilde{\psi}_k(K,\lambda) := \begin{cases} \psi_k(K,\lambda), & k \leqslant K-1, \\ 0, & k \geqslant K. \end{cases}$$

Then $(\widetilde{\psi}(K,\lambda))_{K=k_0(\lambda)}^\infty = ((\widetilde{\psi}_k(K,\lambda))_{k=0}^\infty)_{K=k_0(\lambda)}^\infty$ is bounded in $\mathscr{H}_A^n$, in the sense that its jth column is bounded in $\mathscr{H}_A$, for each $j \in \{1,\ldots,n\}$. Indeed, (3.2), (3.3), (4.1) guarantee the existence of a (λ-dependent) constant c such that $A_k \leqslant cW_k(\lambda)$, and hence

$$\begin{aligned} \sum_{k=0}^\infty \widetilde{\psi}_k^*(K,\lambda)A_k\widetilde{\psi}_k(K,\lambda) &= \sum_{k=0}^{K-1} \psi_k^*(K,\lambda)A_k\psi_k(K,\lambda) \\ &\leqslant c\sum_{k=0}^{K-1} \psi_k^*(K,\lambda)W_k(\lambda)\psi_k(K,\lambda) \\ &= \frac{c}{2}[\widehat{\psi}_K^*(K,\lambda)\mathscr{U}_{2n}J\widehat{\psi}_K(K,\lambda) - \widehat{\psi}_0^*(K,\lambda)\mathscr{U}_{2n}J\widehat{\psi}_0(K,\lambda)], \end{aligned}$$

where we have used Theorem 2.3 for the last identity. The first boundary term is non-positive by (3.4) and (4.9), and the second is bounded with respect to $K \geqslant k_0(\lambda)$ by (4.9), since $l_K(\lambda) \in D_K(\lambda) \subset D_{k_0(\lambda)}(\lambda)$, which is bounded by

Theorem 3.2(ii). This establishes the boundedness of the above sequence. Hence, it has a subsequence $(\widetilde{\psi}(K_m,\lambda))_{m=0}^{\infty}$ which converges weakly in $\mathscr{H}_A^n$ to some $F \in \mathscr{H}_A^n$, so that for every $g \in \mathscr{H}_A^n$,

$$\sum_{k=0}^{\infty} g_k^* A_k \widetilde{\psi}_k(K_m,\lambda) \to \sum_{k=0}^{\infty} g_k^* A_k F_k \quad (m \to \infty). \tag{4.13}$$

Moreover, since $(l_{K_m}(\lambda))_{m=0}^{\infty}$ is bounded in $\mathbb{C}^{n,n}$, it has a convergent subsequence, which we again denote by $\{l_{K_m}(\lambda)\}_{m=0}^{\infty}$. By (4.9) and Theorem 3.2, $M(\lambda) := \lim_{m\to\infty} l_{K_m}(\lambda) \in D_\infty(\lambda)$.

Hence, by (4.5) $\widetilde{\psi}_k(K_m,\lambda)$ converges to $\psi_k(\lambda)$ $(k \in \mathbb{N}_0)$. As the weak convergence implies pointwise convergence, we have $F = \psi(\lambda)$. Thus, choosing $g := (\zeta_k(\mu))_{k\in\mathbb{N}_0}$ in (4.13) (note that $g \in \mathscr{H}_A^n$ because of Remarks (1) and (3) at the end of Section 3),

$$\sum_{k=0}^{K_m-1} \zeta_k^*(\mu) A_k \psi_k(K_m,\lambda) \to \sum_{k=0}^{\infty} \zeta_k^*(\mu) A_k \psi_k(\lambda) \quad (m \to \infty).$$

In a completely analogous way, we can by successive choice of subsequences extract a sequence, which we again denote by $\{K_m\}_{m=0}^{\infty}$, such that

$$\sum_{k=0}^{K_m-1} \zeta_k^*(K_m,\lambda) A_k \psi_k(\mu) \to \sum_{k=0}^{\infty} \zeta_k^*(\lambda) A_k \psi_k(\mu) \quad (m \to \infty).$$

Hence, passing to the limit in (4.12) along the subsequence, we obtain (4.2). Moreover, $l_{K_m}(\mu) = M_0$ $(m \in \mathbb{N}_0)$ implies $M(\mu) = M_0$.

Finally, to show (4.3), we use Theorem 2.3 again to find

$$\widehat{\zeta}_n^*(\mu) J \widehat{\psi}_n(\lambda) + M(\lambda) - M_0 = (\lambda - \mu) \sum_{k=0}^{n-1} \zeta_k^*(\mu) A_k \psi_k(\lambda).$$

The right-hand side converges as $n \to \infty$ since each column of $\zeta_k(\mu)$ and $\psi_k(\lambda)$ is A-square summable. This establishes the existence of the limit of $\widehat{\zeta}_n^*(\mu) J \ \widehat{\psi}_n(\lambda)$ as $n \to \infty$, and comparison with (4.2) provides $\lim_{n\to\infty} \widehat{\zeta}_n^*(\mu) J \ \widehat{\psi}_n(\lambda) = 0$. $\square$

Define the difference expressions $\mathscr{L}$ and $\mathscr{L}^+$ as follows:

$$(\mathscr{L}y)_k := \begin{cases} A_k^{-1}(J\Delta - B_k)y_k, & k \geqslant 1, \\ A_0^{-1}\left[\begin{pmatrix} 0_n & -I_n \\ \Delta_+ I_n & 0_n \end{pmatrix} - B_0\right] y_0, & k = 0, \end{cases}$$

$$(\mathscr{L}^+ z)_k := \begin{cases} A_k^{-1}(J\Delta - B_k^*)z_k, & k \geqslant 1, \\ A_0^{-1}\left[\begin{pmatrix} 0_n & -I_n \\ \Delta_+ I_n & 0_n \end{pmatrix} - B_0^*\right] z_0, & k = 0. \end{cases}$$

Note that $(\mathscr{L}y)_0$ is given by the expression $(\mathscr{L}y)_k$ with $k = 0$ if we take $v_{-1} := 0$; similarly for $(\mathscr{L}^+ z)_0$. It is easy to check that $\mathscr{L}^+$ is the formal adjoint of $\mathscr{L}$ in the sense that

$$(z, \mathscr{L}y)_A = (\mathscr{L}^+ z, y)_A$$

holds if either y or z is a finite sequence.

Remark. Note that

$$(\mathscr{L} - \lambda)\theta_k = 0 \quad \text{for all } k \geqslant 1,$$

$$(\mathscr{L} - \lambda)\phi_k = 0 \quad \text{for all } k \in \mathbb{N}_0$$

$$(\mathscr{L} - \lambda)\psi_k = 0 \quad \text{for all } k \geqslant 1,$$

$$(\mathscr{L} - \lambda)\psi_k \neq 0 \quad \text{for } k = 0.$$

This means that any solution of $(\mathscr{L} - \lambda)y_k = 0$, for all $k \in \mathbb{N}_0$, takes the form $y_k = \phi_k(\lambda)c$, with $c \in \mathbb{C}^n$.
Similarly, $(\mathscr{L}^+ - \overline{\lambda})\chi_0 = 0$ but $(\mathscr{L}^+ - \overline{\lambda})\eta_0 \neq 0$ and $(\mathscr{L}^+ - \overline{\lambda})\zeta_0 \neq 0$.

Now fix any $\mu \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ and $M_0 \in D_\infty(\mu)$. Then, we can define the difference operators L_μ and $\widetilde{L}_\mu$ by setting

$$D(L_\mu) := \left\{ y \in \mathscr{H}_A : \mathscr{L}y \in \mathscr{H}_A \text{ and } \lim_{k\to\infty} \widehat{\zeta}_k^*(\mu) J \widehat{y}_k = 0 \right\},$$

$$L_\mu y := \mathscr{L}y \quad (y \in D(L_\mu))$$

and

$$D(\widetilde{L}_\mu) := \left\{ z \in \mathscr{H}_A : \mathscr{L}^+ z \in \mathscr{H}_A \text{ and } \lim_{k\to\infty} \widehat{\psi}_k^*(\mu) J \widehat{z}_k = 0 \right\},$$

$$\widetilde{L}_\mu z := \mathscr{L}^+ z \quad (z \in D(\widetilde{L}_\mu)).$$

If there is a point $\mu \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ such that $D_\infty(\mu)$ has only one element, L_μ and $\widetilde{L}_\mu$ are unique.

5. The resolvents

We now proceed to study the properties of the operators L_μ and $\widetilde{L}_\mu$ defined in the preceding section, in particular constructing their resolvents. To this end, we first consider the inhomogeneous difference equation, formally $\left(\mathscr{L}_\mu - \lambda\right) y = f$, calculating its Green function by a variant of the variation of constants method. This yields a formal resolvent operator R_λ (Lemma 5.1). We then show that this operator is bounded on $\mathscr{H}_A$ (Theorem 5.3), using an analogue of Fatou's Lemma for series. In fact, R_μ has the properties of the inverse operator of $L_\mu - \mu$ (Lemmas 5.4 and 5.5), and finally we prove that $\Lambda(\lambda_0, \mathscr{U}_{2n})$ is part of the resolvent set of L_μ, and R_λ is its resolvent operator (Theorem 5.7). Furthermore, $\widetilde{L}_\mu$ is the adjoint of L_μ (Lemma 5.6).
The inhomogeneous equation corresponding to (1.3) takes the form

$$J\Delta y_k = (\lambda A_k + B_k)y_k + A_k f_k, \quad k \in \mathbb{N}_0. \tag{5.1}$$

Setting $y_k = \binom{u_k}{v_k}$ and $f_k = \begin{pmatrix} f_k^{(1)} \\ f_k^{(2)} \end{pmatrix}$ with $u_k, v_k, f_k^{(1)}, f_k^{(2)} \in \mathbb{C}^n$, this equation takes the form

$$\begin{pmatrix} v_{k-1} - v_k \\ u_{k+1} - u_k \end{pmatrix} = \begin{pmatrix} (\lambda A_k^{(1)} + B_k^{(1)})u_k + B_k^{(2)} v_k + A_k^{(1)} f_k^{(1)} \\ B_k^{(3)} u_k + (\lambda A_k^{(2)} + B_k^{(4)}) v_k + A_k^{(2)} f_k^{(2)} \end{pmatrix}.$$

Thus

$$v_k = (I_n + B_k^{(2)})^{-1}[-(\lambda A_k^{(1)} + B_k^{(1)})u_k + v_{k-1} - A_k^{(1)} f_k^{(1)}].$$

A solution y_k of (5.1) is related to $\widehat{y}_k = \binom{u_k}{v_{k-1}}$ via

$$y_k = H_k \widehat{y}_k + N_k A_k f_k, \quad k \in \mathbb{N}_0, \tag{5.2}$$

where H_k is given in (2.2) and

$$N_k = \begin{pmatrix} 0_n & 0_n \\ -(I_n + B_k^{(2)})^{-1} & 0_n \end{pmatrix}.$$

Since $(\lambda A_k + B_k)N_k + I_{2n} = \widetilde{H}_k^*$, Eq. (5.1) is equivalent to

$$J\varDelta_+\widehat{y}_k = (\lambda A_k + B_k)H_k\widehat{y}_k + \widetilde{H}_k^* A_k f_k \quad (k \in \mathbb{N}_0). \tag{5.3}$$

Similarly, for the adjoint problem to (5.1),

$$J\Delta z_k = (\overline{\lambda} A_k + B_k^*)z_k + A_k f_k \quad (k \in \mathbb{N}_0), \tag{5.4}$$

we have $z_k = \widetilde{H}_k\widehat{z}_{k+1} + N_k^* A_k f_k$, $k \in \mathbb{N}_0$.

Therefore, observing that $(\overline{\lambda} A_k + B_k^*)N_k^* + I_{2n} = H_k^*$, $k \in \mathbb{N}_0$, (5.4) is equivalent to $J\varDelta_+\widehat{z}_k = H_k^*(\overline{\lambda} A_k + B_k^*)\widehat{z}_{k+1} + H_k^* A_k f_k$, $k \in \mathbb{N}_0$.

Define

$$G_{k,j}(\lambda) := \begin{cases} \psi_k(\lambda)\chi_j^*(\lambda), & 0 \leqslant j < k < \infty, \\ \phi_k(\lambda)\zeta_j^*(\lambda) + N_k\delta_{k,j}, & 0 \leqslant k \leqslant j < \infty, \end{cases}$$

$$\begin{aligned} \widetilde{G}_{k,j}(\lambda) &:= \begin{cases} \chi_k(\lambda)\psi_j^*(\lambda), & 0 \leqslant k < j < \infty \\ \zeta_k(\lambda)\phi_j^*(\lambda) + N_k^*\delta_{k,j}, & 0 \leqslant j \leqslant k < \infty \end{cases} \\ &= G_{j,k}^*(\lambda). \end{aligned}$$

We shall show in Lemma 5.1 below that $G_{k,j}, \widetilde{G}_{k,j}$ are the Green's matrices of (5.1) and (5.4), respectively. For $f = \{f_k\}_{k=0}^{\infty} \in \mathscr{H}_A$, let

$$(R_\lambda f)_k := \sum_{j=0}^{\infty} G_{k,j}(\lambda)A_j f_j,$$

$$\left(\widetilde{R}_\lambda f\right)_k := \sum_{j=0}^{\infty} \widetilde{G}_{k,j}(\lambda)A_j f_j.$$

Lemma 5.1. *Let $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n}) \cap \widetilde{\Lambda}(\lambda_0, \mathscr{U}_{2n}^{-1})$, $f \in \mathscr{H}_A$. Then $R_\lambda f$ is a solution of* (5.1) *and satisfies $(R_\lambda f)_{-1}^{(2)} = 0_n$. In particular, it satisfies the boundary conditions $\widehat{\chi}_0^*\ J(\widehat{R}_\lambda f)_0 = 0$ and $\lim_{k\to\infty} \widehat{\zeta}_k^*(\lambda)\ J(\widehat{R}_\lambda f)_k = 0$, where*

$$(\widehat{R}_\lambda f)_k = \widehat{\psi}_k(\lambda)\sum_{j=0}^{k-1} \chi_j^*(\lambda)A_j f_j + \widehat{\phi}_k(\lambda)\sum_{j=k}^{\infty} \zeta_j^*(\lambda)A_j f_j \quad (k \in \mathbb{N}_0). \tag{5.5}$$

Remark. In particular, $(\mathscr{L} - \lambda)(R_\lambda f) = f$, so R_λ is a candidate for a resolvent for a realisation of $\mathscr{L}$. Furthermore, note that $\widehat{(R_\lambda f)}_k = (\widehat{R}_\lambda f)_k = H_k^{-1}[(R_\lambda f)_k - N_k A_k f_k]$, $k \in \mathbb{N}_0$ using (5.2).

Proof of Lemma 5.1. Applying the variation of parameter method to (5.3), we are looking for a $\mathbb{C}^{2n,2n}$ matrix solution $\widehat{U}_k$ in the form $\widehat{U}_k = \widehat{Y}_k\ \mathscr{M}_k$, where $\mathscr{M}_k$ is an unknown $\mathbb{C}^{2n,2n}$ matrix. Then

$$\varDelta_+\widehat{U}_k = \widehat{Y}_{k+1}(\varDelta_+\mathscr{M}_k) + (\varDelta_+\widehat{Y}_k)\mathscr{M}_k.$$

Applying J to both sides and using the fact that $\widehat{Y}_k$ is a solution of (2.3), we find $\widetilde{H}_k^* A_k f_k = J\ \widehat{Y}_{k+1}\ \varDelta_+\mathscr{M}_k$. Hence by (2.5) and Lemma 2.4

$$\begin{aligned} \varDelta_+\mathscr{M}_k &= -\widehat{Y}_{k+1}^{-1} J\widetilde{H}_k^* A_k f_k \\ &= -J\widehat{Z}_{k+1}^*\widetilde{H}_k^* A_k f_k \\ &= \begin{pmatrix} 0_n & I_n \\ 0_n & M(\lambda) \end{pmatrix} Z_k^* A_k f_k - \begin{pmatrix} 0_n & 0_n \\ I_n & M(\lambda) \end{pmatrix} Z_k^* A_k f_k. \end{aligned}$$

Thus, up to addition of a constant matrix, we have

$$\mathscr{M}_k = \sum_{j=0}^{k-1} \begin{pmatrix} 0_n & I_n \\ 0_n & M \end{pmatrix} Z_j^* A_j f_j + \sum_{j=k}^{\infty} \begin{pmatrix} 0_n & 0_n \\ I_n & M \end{pmatrix} Z_j^* A_j f_j,$$

and consequently

$$\begin{aligned} \widehat{U}_k &= \sum_{j=0}^{k-1} (\widehat{\theta}_k | \widehat{\phi}_k) \begin{pmatrix} \chi_j^* \\ M\chi_j^* \end{pmatrix} \quad A_j f_i + \sum_{j=k}^{\infty} (\widehat{\theta}_k | \widehat{\phi}_k) \begin{pmatrix} 0_n \\ \eta_j^* + M\chi_j^* \end{pmatrix} A_j f_j \\ &= \widehat{\psi}_k \sum_{j=0}^{k-1} \chi_j^* A_j f_j + \widehat{\phi}_k \sum_{j=k}^{\infty} \zeta_j^* A_j f_j. \end{aligned}$$

Hence, by (5.2)

$$\begin{aligned} U_k &= H_k \widehat{U}_k + N_k A_k f_k = \psi_k \sum_{j=0}^{k-1} \chi_j^* A_j f_j + \phi_k \sum_{j=k}^{\infty} \zeta_j^* A_j f_j + N_k A_k f_k \\ &= (R_\lambda f)_k. \end{aligned}$$

It is not difficult to verify that U_k is a solution of (5.1).

Now

$$\widehat{(R_\lambda f)}_0 = (\widehat{R}_\lambda f)_0 = \widehat{\phi}_0(\lambda) \sum_{j=0}^{\infty} \zeta_j^*(\lambda) A_j f_j = \begin{pmatrix} -\sum_{j=0}^{\infty} \zeta_j^*(\lambda) A_j f_j \\ 0_n \end{pmatrix}$$

which implies that $(R_\lambda f)_{-1}^{(2)} = 0_n$. The identity

$$\widehat{\chi}_0^* J (\widehat{R}_\lambda f)_0 = \widehat{\chi}_0^* J \widehat{\phi}_0 \sum_{j=0}^{\infty} \zeta_j^* A_j f_j = 0$$

follows directly from the initial values of χ and ϕ. The sum is convergent since each column of ζ is A-square summable and $f \in \mathscr{H}_A$. Moreover, since $\widehat{\psi}_k = \widehat{Y}_k \binom{I_n}{M}$ and $\widehat{\zeta}_k = \widehat{Z}_k \binom{I_n}{M^*}$, we find using Lemma 2.4 that

$$\widehat{\zeta}_k^*(\lambda) J \widehat{\psi}_k(\lambda) = (I_n | M) \widehat{Z}_k^*(\lambda) J \widehat{Y}_k(\lambda) \begin{pmatrix} I_n \\ M \end{pmatrix} = 0_n$$

and

$$\widehat{\zeta}_k^*(\lambda) J \widehat{\phi}_k(\lambda) = (I_n | M) \widehat{Z}_k^*(\lambda) J \widehat{Y}_k(\lambda) \begin{pmatrix} 0_n \\ I_n \end{pmatrix} = -I_n. \tag{5.6}$$

Hence $\lim_{k\to\infty} \widehat{\zeta}_k^*(\lambda)\ J(\widehat{R}_\lambda f)_k = -\lim_{k\to\infty} \sum_{j=k}^{\infty} \zeta_j^*\ A_j f_j = 0$. $\square$

For the adjoint problem (5.4), we obtain similarly

$$J\Delta(\widetilde{R}_\lambda f)_k = (\bar{\lambda} A_k + B_k^*)(\widetilde{R}_\lambda f)_k + A_k f_k,$$

with boundary conditions $\widehat{\phi}_0^* J(\widehat{\widetilde{R}}_\lambda f)_0 = 0$ and $\lim_{k\to\infty} \widehat{\psi}_k^*(\lambda) J(\widehat{\widetilde{R}}_\lambda f)_k = 0$.

Lemma 5.2 (*Fatou's lemma for series*). *For each $K \in \mathbb{N}_0$, let $\{f_{K,k}\}_{k=0}^{\infty}$ be a non-negative summable sequence such that the pointwise limit*

$$f_k := \lim_{K\to\infty} f_{K,k}$$

exists for all $k \in \mathbb{N}_0$. *Then,*

$$\sum_{k=0}^{\infty} f_k \leqslant \liminf_{K\to\infty} \sum_{k=0}^{\infty} f_{K,k}.$$

In particular, f_k *is summable if the right-hand side is finite.*

Proof. For any $n \in \mathbb{N}_0$, we have that $\sum_{k=0}^{n} f_{K,k} \leqslant \sum_{k=0}^{\infty} f_{K,k}$, so

$$\sum_{k=0}^{n} f_k = \sum_{k=0}^{n} \lim_{K\to\infty} f_{K,k} = \lim_{K\to\infty} \sum_{k=0}^{n} f_{K,k} \leqslant \liminf_{K\to\infty} \sum_{k=0}^{\infty} f_{K,k}. \qquad \square$$

Theorem 5.3. *Let* $f = \{f_k\}_{k=0}^{\infty} \in \mathscr{H}_A$, $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n}) \cap \widetilde{\Lambda}(\lambda_0, \mathscr{U}_{2n}^{-1})$. *Then with* $\Phi = R_\lambda f$ *and* $\widetilde{A}_k = \mathscr{U}_{2n} A_k \mathscr{U}_{2n}^*$,

$$\|\Phi\|_{W(\lambda_0)}^2 + (\delta - \varepsilon)\|\Phi\|_{\widetilde{A}}^2 \leqslant \frac{1}{4\varepsilon}\|f\|_A^2$$

for any $0 < \varepsilon < \delta$, *with* $\delta = \delta(\lambda)$ *as in* (3.3), *and*

$$\|\Phi\|_{\widetilde{A}} \leqslant \frac{1}{\delta}\|f\|_A.$$

In particular, since $\widetilde{A}_k \asymp A_k$, R_λ *is bounded on* $\mathscr{H}_A$.

Proof. Let

$$f_{K,k} = \begin{cases} f_k, & k < K, \\ 0, & k \geqslant K \end{cases}$$

and $\Phi_{K,k} = (R_\lambda f_K)_k$. Then in analogy to (3.1)

$$2\sum_{k=0}^{K-1} \Phi_{K,k}^* W_k(\lambda)\Phi_{K,k} = \widehat{\Phi}_{K,k}^* \mathscr{U}_{2n} J \widehat{\Phi}_{K,k}\Bigg|_{k=0}^{K} - 2\sum_{k=0}^{K-1} \mathrm{Re}[\Phi_{K,k}^* \mathscr{U}_{2n} A_k f_{K,k}]. \tag{5.7}$$

Now, using the Cauchy–Schwartz inequality for $\mathbb{C}^n$,

$$\begin{aligned} |\mathrm{Re}[\Phi_{K,k}^* \mathscr{U}_{2n} A_k f_{K,k}]| &\leqslant |\Phi_{K,k}^* \mathscr{U}_{2n} A_k f_{K,k}| = |\langle A_k f_{K,k}, \mathscr{U}_{2n}^* \Phi_{K,k}\rangle| \\ &\leqslant \langle A_k f_{K,k}, f_{K,k}\rangle^{1/2} \langle A_k \mathscr{U}_{2n}^* \Phi_{K,k}, \mathscr{U}_{2n}^* \Phi_{K,k}\rangle^{1/2} \\ &\leqslant (f_{K,k}^* A_k f_{K,k})^{1/2} (\Phi_{K,k}^* \widetilde{A}_k \Phi_{K,k})^{1/2}, \end{aligned}$$

and as in the proof of Theorem 5.1 in [4] one can show that $\widehat{\Phi}_{K,0}^* \mathscr{U}_{2n} J\, \widehat{\Phi}_{K,0} = 0$ and $\widehat{\Phi}_{K,K}^* \mathscr{U}_{2n} J\, \widehat{\Phi}_{K,K} \leqslant 0$. Hence we obtain from (5.7) that

$$\begin{aligned} \sum_{k=0}^{K-1} \Phi_{K,k}^* W_k(\lambda)\Phi_{K,k} &\leqslant \sum_{k=0}^{K-1} [(f_{K,k}^* A_k f_{K,k})^{1/2} (\Phi_{K,k}^* \widetilde{A}_k \Phi_{K,k})^{1/2}] \\ &\leqslant \left(\sum_{k=0}^{K-1} f_{K,k}^* A_k f_{K,k}\right)^{1/2} \left(\sum_{k=0}^{K-1} \Phi_{K,k}^* \widetilde{A}_k \Phi_{K,k}\right)^{1/2} \\ &\leqslant \frac{1}{4\varepsilon}\|f\|_A^2 + \varepsilon \sum_{k=0}^{K-1} \Phi_{K,k}^* \widetilde{A}_k \Phi_{K,k} \quad \text{for any } \varepsilon > 0. \end{aligned}$$

By (3.3), we have for $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n}) \cap \widetilde{\Lambda}(\lambda_0, \mathscr{U}_{2n}^{-1})$,

$$\begin{aligned}\Phi_{K,k}^*(W_k(\lambda) - W_k(\lambda_0))\Phi_{K,k} &= \Phi_{K,k}^*\mathrm{Re}[(\lambda-\lambda_0)\mathscr{U}_{2n}A_k]\Phi_{K,k}\\ &\geqslant \delta\Phi_{K,k}^*\widetilde{A}_k\Phi_{K,k}.\end{aligned}$$

Hence,

$$\sum_{k=0}^{K-1} \Phi_{K,k}^* W_k(\lambda_0)\Phi_{K,k} + (\delta-\varepsilon)\sum_{k=0}^{K-1} \Phi_{K,k}^*\widetilde{A}_k\Phi_{K,k} \leqslant \frac{1}{4\varepsilon}\|f\|_A^2.$$

Let

$$\widetilde{\Phi}_{K,k} = \begin{cases} \Phi_{K,k}, & k<K,\\ 0, & k \geqslant K,\end{cases}$$

then

$$\sum_{k=0}^{\infty} \widetilde{\Phi}_{K,k}^* W_k(\lambda_0)\widetilde{\Phi}_{K,k} + (\delta-\varepsilon)\sum_{k=0}^{\infty} \widetilde{\Phi}_{K,k}^*\widetilde{A}_k\widetilde{\Phi}_{K,k} \leqslant \frac{1}{4\varepsilon}\|f\|_A^2.$$

As $K \to \infty$, $\widetilde{\Phi}_{K,k} \to \Phi_k$ pointwise for $k \in \mathbb{N}_0$. Thus, for all $k \in \mathbb{N}_0$,

$$\lim_{K\to\infty} \widetilde{\Phi}_{K,k}^* W_k(\lambda_0)\widetilde{\Phi}_{K,k} = \Phi_k^* W_k(\lambda_0)\Phi_k \quad \text{and} \quad \lim_{K\to\infty} \widetilde{\Phi}_{K,k}^*\widetilde{A}_k\widetilde{\Phi}_{K,k} = \Phi_k^* A_k\Phi_k.$$

Hence, if $\varepsilon < \delta$, it follows from Lemma 5.2 that

$$\begin{aligned}&\|\Phi\|_{W(\lambda_0)}^2 + (\delta-\varepsilon)\|\Phi\|_{\widetilde{A}}^2\\ &\quad\leqslant \liminf_{K\to\infty}\left[\sum_{k=0}^{\infty} \widetilde{\Phi}_{K,k}^* W_k(\lambda_0)\widetilde{\Phi}_{K,k} + (\delta-\varepsilon)\sum_{k=0}^{\infty} \widetilde{\Phi}_{K,k}^*\widetilde{A}_k\widetilde{\Phi}_{K,k}\right]\\ &\quad\leqslant \frac{1}{4\varepsilon}\|f\|_A^2.\end{aligned}$$

The remaining statements follow by choosing $\varepsilon = \frac{1}{2}\delta$. $\square$

Lemma 5.4. *Let $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})$. Then R_λ is an injective linear operator defined on $\mathscr{H}_A$. Moreover, for any $\mu \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ we have* range $R_\mu \subset D(L_\mu)$ *and $(L_\mu - \mu)R_\mu f = f$, for all $f \in \mathscr{H}_A$. A corresponding statement holds for $\widetilde{R}_\mu$.*

Proof. Let $f \in \mathscr{H}_A$. As seen in Lemma 5.1,

$$(\mathscr{L} - \lambda)(R_\lambda f) = f \tag{5.8}$$

and

$$\lim_{k\to\infty} \widehat{\zeta}_k^*(\lambda) J(\widehat{R}_\lambda f)_k = 0. \tag{5.9}$$

It is clear from its definition that R_λ is linear. Moreover, R_λ is injective, since if $f \in \mathscr{H}_A$ is such that $R_\lambda f = 0$, it implies that $\mathscr{L}(R_\lambda f) = 0$, and hence by (5.8) also $f = 0$.

Now let $\mu \in \Lambda(\lambda_0, \mathscr{U}_{2n})$ be fixed and let $h \in$ range R_μ. Then $h = R_\mu f$ for some $f \in \mathscr{H}_A$. By Theorem 5.3, we conclude that $h \in \mathscr{H}_A$. but then $\mathscr{L}h = f - \lambda h \in \mathscr{H}_A$ as well, and (5.9) with $\lambda = \mu$ implies that $h \in D(L_\mu)$. Clearly $(L_\mu - \mu)h = f$. $\square$

Lemma 5.5. *Denoting the resolvent sets of L_μ and $\widetilde{L}_\mu$ by $\rho(L_\mu)$ and $\rho(\widetilde{L}_\mu)$, respectively, we have $\mu \in \rho(L_\mu)$, $\overline{\mu} \in \rho(\widetilde{L}_\mu)$,* range $R_\mu = D(L_\mu)$, range $\widetilde{R}_\mu = D(\widetilde{L}_\mu)$, *$(L_\mu - \mu)^{-1} = R_\mu$ and $(\widetilde{L}_\mu - \overline{\mu})^{-1} = \widetilde{R}_\mu$. Moreover, L_μ and $\widetilde{L}_\mu$ are closed.*

Proof. $L_\mu - \mu$ is injective, since $y \in D(L_\mu)$, $(L_\mu - \mu)y = 0$ imply, by the remark at the end of Section 4, that $y = \phi(\mu)v$ for some $v \in \mathbb{C}^n$. By (5.6) and the boundary condition for y for $k \to \infty$, it follows that $y = 0$.

Lemma 5.4 implies that range$(L_\mu - \mu) = \mathscr{H}_A$, and hence that $(L_\mu - \mu)^{-1}$ is defined on $\mathscr{H}_A$. Moreover $D(L_\mu) \subset$ range R_μ. Indeed, let $y \in D(L_\mu)$ and take $f = (L_\mu - \mu)y \in \mathscr{H}_A$. Then $R_\mu(L_\mu - \mu)y = (L_\mu - \mu)^{-1}(L_\mu - \mu)y = y$, so $y \in$ range R_μ. Thus $(L_\mu - \mu)^{-1} = R_\mu$, so $(L_\mu - \mu)^{-1}$ is bounded by Theorem 5.3, whence $\mu \in \rho(L_\mu)$. In particular, $(L_\mu - \mu)$ and hence L_μ are closed (cf. [8, Theorem 5.8]). The statements for $\widetilde{L}_\mu$ and $\widetilde{R}_\mu$ follow analogously. $\square$

Lemma 5.6. *The space $D(L_\mu)$ is dense in $\mathscr{H}_A$. Also $\widetilde{L}_\mu = L_\mu^*$, the adjoint of L_μ.*

Proof. By the boundedness of R_μ and $\widetilde{R}_\mu$ on $\mathscr{H}_A$ shown in Theorem 5.3, we have $\widetilde{R}_\mu = R_\mu^*$, the $(.,.)_A$-adjoint of R_μ. Thus, Lemma 5.4 shows that R_μ^* is injective, so $D(L_\mu) =$ range R_μ is dense in $\mathscr{H}_A$. Moreover,

$$(\widetilde{L}_\mu - \overline{\mu})^{-1} = [(L_\mu - \mu)^{-1}]^* = (L_\mu^* - \overline{\mu})^{-1}$$

whence $\widetilde{L}_\mu = L_\mu^*$. $\square$

Theorem 5.7. *Let $\mu \in \Lambda(\lambda_0, \mathscr{U}_{2n})$. Then $\Lambda(\lambda_0, \mathscr{U}_{2n}) \subset \rho(L_\mu)$ and*

$$(L_\mu - \lambda)^{-1} = R_\lambda \quad (\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})).$$

A corresponding statement holds for $\widetilde{R}_\lambda$ and $\widetilde{L}_\mu$.

Proof. Let $\lambda \in \Lambda(\lambda_0, \mathscr{U}_{2n})$. For all $f \in \mathscr{H}_A$ we have by Lemma 5.1 that

$$(\mathscr{L} - \lambda)R_\lambda f = f$$

which shows that $\mathscr{L} R_\lambda f \in \mathscr{H}_A$, since $R_\lambda f \in \mathscr{H}_A$ follows from Theorem 5.3. Next we prove that

$$\lim_{k\to\infty} \widehat{\zeta}_k^*(\mu) J(\widehat{R}_\lambda f)_k = 0. \tag{5.10}$$

This is clear for a finite sequence $\{f_k\}_{k=0}^\infty$, with $f_k = 0$ for $k \geqslant K$, since (5.5) gives

$$(\widehat{R}_\lambda f)_k = \widehat{\psi}_k(\lambda) \sum_{j=0}^{K-1} \chi_j^*(\lambda) A_j f_j \quad (k \geqslant K)$$

whence (4.3) in Theorem 4.1 implies that

$$\lim_{k\to\infty} \widehat{\zeta}_k^*(\mu) J(\widehat{R}_\lambda f)_k = \lim_{k\to\infty} \widehat{\zeta}_k^*(\mu) J \widehat{\psi}_k(\lambda) \sum_{j=0}^{K-1} \chi_j^*(\lambda) A_j f_j = 0.$$

To obtain (5.10) for general $f \in \mathscr{H}_A$, we define, for each $m \in \mathbb{N}_0$, a finite sequence $f^{(m)}$ in $\mathscr{H}_A$, where

$$f_k^{(m)} = \begin{cases} f_k, & k < m, \\ 0, & k \geqslant m. \end{cases}$$

Then $f^{(m)} \to f$ in $\mathscr{H}_A$ as $m \to \infty$. It follows from (5.5) and (5.6) that

$$\begin{aligned} \widehat{\zeta}_0^*(\mu) J(\widehat{R}_\lambda(f - f^{(m)}))_0 &= \widehat{\zeta}_0^*(\mu) J \widehat{\phi}_0(\lambda) \sum_{j=0}^{\infty} \zeta_j^*(\lambda) A_j (f - f^{(m)})_j \\ &= ((f^{(m)} - f), \zeta(\lambda))_A. \end{aligned}$$

Hence, using Theorem 2.3, and the facts that $R_\lambda(f - f^{(m)})$ is a solution of the inhomogeneous (5.1), ζ is a solution of (2.4),

$$\begin{aligned}\widehat{\zeta}_k^*(\mu)J(\widehat{R}_\lambda f)_k &= \widehat{\zeta}_k^*(\mu)J(\widehat{R}_\lambda f^{(m)})_k + \widehat{\zeta}_0^*(\mu)J(\widehat{R}_\lambda(f - f^{(m)}))_0 \\ &\quad + \sum_{j=0}^{k-1}[\zeta_j^*(\mu)J\Delta(R_\lambda(f - f^{(m)}))_j - (J\Delta\zeta_j(\mu))^*(R_\lambda(f - f^{(m)}))_j] \\ &= \widehat{\zeta}_k^*(\mu)J(\widehat{R}_\lambda f^{(m)})_k + ((f^{(m)} - f), \zeta(\lambda))_A \\ &\quad + \sum_{j=0}^{k-1}\zeta_j^*(\mu)A_j[(\lambda-\mu)(R_\lambda(f - f^{(m)}))_j + (f - f^{(m)})_j].\end{aligned}$$

As seen above, the first term on the right-hand side tends to 0 as $k \to \infty$. Moreover, the sum on the right-hand side converges as $k \to \infty$ since the columns of $\zeta(\mu)$ are A-square summable and $R_\lambda(f - f^{(m)})$, $(f - f^{(m)}) \in \mathscr{H}_A$. Thus,

$$\begin{aligned}\lim_{k\to\infty} \widehat{\zeta}_k^*(\mu)J(\widehat{R}_\lambda f)_k &= ((f^{(m)} - f), \zeta(\lambda))_A + ((\lambda-\mu)R_\lambda(f - f^{(m)}) + (f - f^{(m)}), \zeta(\mu))_A \\ &\leqslant \|f - f^{(m)}\|_A(\|\zeta(\lambda)\|_A + \|\zeta(\mu)\|_A) + |\lambda-\mu|\|R_\lambda\|\|f - f^{(m)}\|_A\|\zeta(\mu)\|_A,\end{aligned}$$

where $\|R_\lambda\|$ is the operator norm in $\mathscr{H}_A$. Now (5.10) follows in the limit $m \to \infty$. Thus, we have proved that $R_\lambda f \in D(L_\mu)$ and $(L_\mu - \lambda)R_\lambda f = f$, implying that the range of $(L_\mu - \lambda)$ is $\mathscr{H}_A$. Analogously, the range of $(\widetilde{L}_\mu - \overline{\lambda})$ is $\mathscr{H}_A$. Since $\widetilde{L}_\mu - \overline{\lambda} = (L_\mu - \lambda)^*$ by Lemma 5.6, this implies that $L_\mu - \lambda$ is injective. Consequently, $(L_\mu - \lambda)^{-1} = R_\lambda$, and as R_λ is bounded, we have $\lambda \in \rho(L_\mu)$. □

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Appendix A.

In the case of the classical Weyl's alternative, it is well known that the limit point/limit circle classification of the Sturm–Liouville equation is independent of the spectral parameter $\lambda \in \mathbb{C}$ considered. Thus, if all solutions are square integrable for some λ, this holds true for all $\lambda \in \mathbb{C}$. A rather more complicated situation appears in the case of the general Hamiltonian system. Nevertheless, we are able to give a condition under which A-square summability of all solutions extends from one point λ to the whole complex plane, by using the solution of the inhomogeneous equation calculated in the preceding section in a way similar to [6, Chapter 9, Theorem 2.1].

Theorem A.1. *If all solutions of* (1.3) *and* (2.4) *are A-square summable for some* $\lambda' \in \mathbb{C}$, *and if* $A_j^{1/2}N_j^*A_jN_jA_j^{1/2} \to 0$ $(j \to \infty)$, *then all solutions of* (1.3) *are A-square summable for all* $\lambda \in \mathbb{C}$.

Proof. Any solution of

$$J\Delta y_k = (\lambda A_k + B_k)y_k = (\lambda' A_k + B_k)y_k + (\lambda - \lambda')A_k y_k$$

can be written as

$$y_k(\lambda) = \theta_k(\lambda')D + \phi_k(\lambda')\widetilde{D} + (\lambda - \lambda')\left[\sum_{j=c}^{k-1}(\psi_k(\lambda')\chi_j^*(\lambda') - \phi_k(\lambda')\zeta_j^*(\lambda'))A_j y_j(\lambda) + N_k A_k y_k(\lambda)\right]$$

for some $D, \widetilde{D} \in \mathbb{C}^n$. We abbreviate

$$\|y\|_{A,c}^2(k) = \sum_{j=c}^{k} y_j^* A_j y_j.$$

Since all solutions of (1.3) and (2.4) are A-square summable at λ', there exists a constant C (independent of k) such that the norm $\|.\|_{A,c}$ of each column of θ, ϕ, χ, ζ, ψ is less than or equal to C. Moreover, by choosing c large enough, we can ensure that $C^2 < 1/8n|\lambda - \lambda'|$.

Thus, denoting by $\theta_{j,\rho}$ the ρth column of θ_j, we find

$$\begin{aligned}\|\theta(\lambda')D\|^2_{A,c}(k) &= \sum_{j=c}^{k}\sum_{\rho,\sigma=1}^{n} \overline{D}_\rho(\theta_j^* A_j\theta_j)_{\rho,\sigma}D_\sigma \\ &\leqslant \sum_{\rho,\sigma=1}^{n} |\overline{D}_\rho| \left(\sum_{j=c}^{k} |(\theta_{j,\rho})^* A_j\theta_{j,\sigma}|\right)|D_\sigma| \\ &\leqslant \|\theta_{j,\rho}\|_{A,c}(k)\|\theta_{j,\sigma}\|_{A,c}(k)\sum_{\rho,\sigma=1}^{n}|\overline{D}_\rho||D_\sigma| \\ &\leqslant n^2C^2C_1^2\end{aligned}$$

for some constant C_1 independent of k. Similarly,

$$\|\phi(\lambda')\widetilde{D}\|^2_{A,c}(k) \leqslant n^2C^2C_2^2$$

with a constant C_2 independent of k. We also have

$$\left\|\sum_{j=c}^{.-1}(\psi_.(\lambda')\chi_j^*(\lambda') - \phi_.(\lambda')\zeta_j^*(\lambda'))A_j y_j(\lambda)\right\|^2_{A,c}(k) \leqslant \sum_{K=c}^{k}\sum_{j,l=c}^{K-1}(|F_1|+|F_2|+|F_3|+|F_4|),$$

where

$$\begin{aligned}F_1 &= y_j^*A_j\chi_j(\psi_K^*A_K\psi_K)\chi_l^*A_l y_l,\\ F_2 &= y_j^*A_j\chi_j(\psi_K^*A_K\phi_K)\zeta_l^*A_l y_l,\\ F_3 &= y_j^*A_j\zeta_j(\phi_K^*A_K\psi_K)\chi_l^*A_l y_l,\\ F_4 &= y_j^*A_j\zeta_j(\phi_K^*A_K\phi_K)\zeta_l^*A_l y_l.\end{aligned}$$

Since $K \leqslant k$ it follows, again with $(.)_\rho$ denoting the ρth column, that

$$\begin{aligned}\sum_{K=c}^{k}\sum_{j,l=c}^{K-1}|F_1| &= \sum_{K=c}^{k}\sum_{j,l=c}^{K-1}\sum_{\rho,\sigma=1}^{n}|(y_j^*A_j\chi_j)_\rho||(\psi_K^*A_K\psi_K)_{\rho,\sigma}||(\chi_l^*A_l y_l)^{\mathrm{T}}_\sigma| \\ &= \sum_{\rho,\sigma=1}^{n}\left(\sum_{j=c}^{K-1}|y_j^*A_j\chi_{j,\rho}|\sum_{K=c}^{k}|(\psi_K^*A_K\psi_K)_{\rho,\sigma}|\sum_{l=c}^{K-1}|(\chi_{l,\sigma})^*A_l y_l|\right) \\ &\leqslant C^2\|y\|^2_{A,c}(k)\sum_{\rho,\sigma=1}^{n}\sum_{K=c}^{k}|(\psi_{K,\rho})^*A_K\psi_{K,\sigma}| \\ &\leqslant n^2C^4\|y\|^2_{A,c}(k).\end{aligned}$$

Analogous calculation for F_2, F_3, F_4 yields

$$\left\|\sum_{j=c}^{.-1}(\psi_.(\lambda')\chi_j^*(\lambda') - \phi_.(\lambda')\zeta_j^*(\lambda'))A_j y_j(\lambda)\right\|^2_{A,c}(k) \leqslant 4n^2C^4\|y\|^2_{A,c}(k).$$

Moreover,

$$
\begin{aligned}
\|NAy\|_{A,c}^2(k) &= \sum_{j=c}^{k} y_j^* A_j N_j^* A_j N_j A_j y_j \\
&= \sum_{j=c}^{k} y_j^* A_j^{1/2} (A_j^{1/2} N_j^* A_j N_j A_j^{1/2}) A_j^{1/2} y_j \\
&\leqslant \sum_{j=c}^{k} \|A_j^{1/2} N_j^* A_j N_j A_j^{1/2}\| y_j^* A_j y_j \\
&\leqslant \max_{j \geqslant c} \|A_j^{1/2} N_j^* A_j N_j A_j^{1/2}\| \|y\|_{A,c}^2(k).
\end{aligned}
$$

Thus we conclude that

$$
\|y(\lambda)\|_{A,c}(k) \leqslant nC(C_1 + C_2) + |\lambda - \lambda'| \left(2nC^2 + \max_{j \geqslant c} \sqrt{\|A_j^{1/2} N_j^* A_j N_j A_j^{1/2}\|} \right) \|y(\lambda)\|_{A,c}(k).
$$

If c is chosen large enough so that $|\lambda - \lambda'| \max_{j \geqslant c} \sqrt{\|A_j^{1/2} N_j^* A_j N_j A_j^{1/2}\|} \leqslant \frac{1}{4}$, then $\|y(\lambda)\|_{A,c}(k) \leqslant 2nC(C_1 + C_2)$. Since the right-hand side of this inequality is independent of k, it follows that y is A-square summable and the theorem is proven. □

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